On the Optimality of Exclusion in Multi-dimensional Screening

Paulo Barelli Suren Basov Mauricio Bugarin Ian King *

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Abstract

We extend Armstrong's [1] result on exclusion in multi-dimensional screening models providing support for the view that the result holds true in a large class of models. We first relax some of the strong technical assumptions he imposed and provide alternative sufficient conditions for exclusion not relying on any form of convexity. We then proceed to show that exclusion obtains generically. We illustrate the results with examples and applications.

JEL Codes: C72, D42, D43, D82

Key words: Multi-dimensional screening, exclusion, regulation of a monopoly, involuntary unemployment.

1 Introduction

When considering the problem of screening, where sellers choose a sales mechanism and buyers have private information about their types, it is well known that the techniques used in the multi-dimensional setting are not as straightforward as those in the one-dimensional setting. As a consequence, while we have a host of successful applications with one-dimensional

*Paulo Barelli: University of Rochester, Suren Basov: La Trobe University, Mauricio Bugarin: University of Brasilia, Ian King: University of Melbourne. We are grateful to Mark Armstrong, Steve Williams, Aloisio Araujo, the seminar audiences at SAET 2010, Deakin University and Boston University, especially Hsueh-Ling Huynh. We are also indebted to the editor for numerous useful comments and suggestions, and three anonymous referees for helpful comments.

types, to date we have only a few scattered papers that allow for multi-dimensional types. This is unfortunate because in many, if not most, economic applications multi-dimensional types are needed to capture the basic economics of the environment, and the propositions coming from the one-dimensional case do not necessarily generalize to the multi-dimensional case.[1]

One intriguing result in the theory of multi-dimensional screening comes from Armstrong [1], who shows that a monopolist will find it optimal to not serve some fraction of consumers, even when there is positive surplus associated with those consumers. That is, in settings where consumers might differ in at least two characteristics, monopolists will choose a sales mechanism that excludes a positive measure of consumers. The intuition behind this result is as follows: consider a situation where the monopolist serves all consumers; if she increases the price by $\varepsilon > 0$ she earns extra profits of order $O(\varepsilon)$ on the consumers who still buy the product, but will lose only the consumers whose surplus was below ε; if the dimension of the vector of consumers' taste characteristics is greater than one, the space of such vectors is strictly convex, and the surplus function is quasi-convex, then the measure of the set of the lost consumers is of order $O(\varepsilon^m)$; therefore, it is profitable to increase the price and exclude some consumers. In principle, this result has profound implications across a wide range of economic settings. The general belief that heterogeneity of consumer types is likely to be more than one-dimensional in nature, for many different commodities, and that these types are likely to be private information, underlines the significance of Armstrong's result.[2] Moreover, the intuition provided above seems to be robust, i.e., seems to not depend on particular technical details of the model.

However, Armstrong's [1] result was derived under a relatively strong set of assumptions, which could be seen as limiting its applicability, and subsequent research has identified conditions under which the result does not hold. In particular, in addition to assuming that types belong to a strictly convex and compact body of a finite dimensional space, Armstrong obtains quasi-convexity of the surplus function by assuming that the utility functions of the agents are quasi-linear, homogeneous and convex in their types. Basov [5] refers to this pair of convexity requirements as the joint convexity assumption and argues that, although convexity of utility in types and convexity of the types set separately are not restrictive and can be seen as a choice of parametrization, the joint convexity assumption is technically

[1]See Rochet and Stole [24] and Basov [5] for surveys of the literature.

[2]The type of an economic agent is simply her utility function. If one is agnostic about the preferences and does not want to impose on them any assumptions beyond, perhaps, monotonicity and convexity, then the most natural assumption is that the type is multi-dimensional.

restrictive.

The joint convexity assumption has no empirical foundation and is nonstandard. For instance, the benchmark case of independent types fails joint convexity because the type space is the not strictly convex multi-dimensional box. There is, in general, no theoretical justification for a particular assumption about the curvature of utility functions with respect to types, as opposed to, say, quasi-concavity of utility functions with respect to goods. In the same line, in general there is no justification, other than analytical tractability, for type spaces to be convex, and for utility functions to be homogeneous in types. On the other hand, Armstrong [2], Rochet and Stole [24], Jehiel, Moldovanu and Stacchetti [16], and Severinov and Deneckere [25] found examples outside of these restrictions where the exclusion set is empty. That is, the technical conditions provided by Armstrong [1] cannot simply be dropped.

It turns out that the conditions can be improved upon. In particular, there's no need to assume that utility functions are quasi-linear, homogeneous, or convex in types, or that the type space is convex. In Theorem 1 we establish that exclusion is optimal whenever the utility functions are increasing in types, bounded, and the boundary structure of the type space is of a particular kind.[3] That is, our assumptions on the utility functions are substantially weaker than Armstrong's, but the assumption on type spaces is non-nested with his assumption of a strictly convex type space. We do allow for non-convex type spaces, but the requirement on the boundary structure need not hold for a given strictly convex type space.

Moreover, we show that the counter-examples found in the literature are knife-edge cases. In Theorem 2 we establish that exclusion is generically optimal for a monopolist in the family of models where utility functions are of class C^1 and monotone in types, and types belong to sets of locally finite perimeter. The class of sets of locally finite perimeter is a class of sets that includes all of the examples the authors are aware of in the literature, and we stress, includes type spaces that are nowhere close to being convex. That is, exclusion is generically optimal in a large class of models that contains the model used by Armstrong.

We illustrate the generality of the results with a few examples and two applications, namely the regulation of a monopolist with unknown demand and cost functions, and the emergence of involuntary unemployment as a result of screening by employers. The former application picks up of the analysis in Armstrong [2], where he reviews Lewis and Sapping-

[3]Rightward-slanted diamonds are typical examples of type spaces having boundary structures of the kinds required for Theorem 1, but the class is much larger and includes non-convex sets.

ton [19] and conjectures that exclusion is probably an issue in their analysis. At the time, Armstrong could not prove the point, due to the lack of a more general exclusion result. With our results in hand, we are able to establish that Armstrong's conjecture is generically true. The latter application is a straightforward way of showing that, when workers have multi-dimensional characteristics, it is generically optimal for employers (with market power in the labor market) to not hire all the workers.

In sum, the paper provides evidence to the proposition that private information leads to exclusion in many realistic monopolistic settings. To avoid it, one must either assume that all allowable preferences lie on a one-dimensional continuum, or construct very specific type distributions and preferences.

The remainder of this paper is organized as follows. In Section 2 we present the monopoly problem with consumers with multi-dimensional characteristics, together with assumptions on the underlying parameters under which it is generically optimal to have exclusion. We present the economic and geometric arguments behind our results in Section 3. Section 4 presents examples and applications, illustrating further the genericity of exclusion and the impact of such a fact in economic applications. The proofs of our results are in the Appendix.

2 Monopolistic Screening and Results

Consider a firm with a monopoly over n goods. The tastes of the consumers over these goods are parametrized by a vector $\alpha \in \mathbb{R}^m_+$. The utility of a type α consumer consuming a bundle $x \in \mathbb{R}^n_+$ and paying $t \in \mathbb{R}$ to the firm is

$$v(\alpha, x, t),$$

where v is strictly increasing and strictly concave in x, and strictly decreasing in t. Our focus is not on relaxing the smoothness assumptions on v, so we will assume that v is twice continuously differentiable, with $v_t(\alpha, x, t) \equiv \frac{\partial v(\alpha,x,t)}{\partial t} < 0$ Lipschitz continuous, bounded below and away from zero. Furthermore, we assume that $v_{i,t} \equiv \frac{\partial^2 v}{\partial \alpha_i \partial t} \leq 0$ for all $i = 1, ..., m$.[4] The case of quasilinear preferences $v(\alpha, x, t) = \nu(\alpha, x) - t$ is a special case with $v_t = -1$ and $v_{i,t} = 0$ for all $i = 1, ..., m$.

[4]The economic rationale for this assumption is that if it did not hold (that is, if $v_{i,t} \equiv \frac{\partial^2 v}{\partial \alpha_i \partial t} > 0$ for some i) we would have higher types less price sensitive, which would give an extra incentive for the monopolist to charge them higher prices; we assume this case away to concentrate on price discrimination alone.

The total cost of producing bundle x is given by $c(x)$, where c is a convex function (possibly linear). The set of feasible production vectors is denoted by $X \subset \mathbb{R}^n_+$. The firm is not able to observe the consumer's type, but has prior beliefs over the distribution of types, described by the density function $f(\alpha)$, with compact support $\text{supp}(f) = \overline{\Omega}$, where $\Omega \subset \mathbb{R}^m$ is the space of types, and $\overline{\Omega}$ is its closure. We assume that $\Omega \subset U$ is a bounded open set with *locally finite perimeter* in the open set U, and that f is Lipschitz continuous.[5] Intuitively, a set has locally finite perimeter if its characteristic function is a function of bounded variation, hence it is a large class of open sets that includes the class of open convex sets as a very small subclass.[6] We assume that $v(\cdot, x, t)$ can be extended by continuity to $\overline{\Omega}$. Consumers have an outside option of value $s_0(\alpha)$, which is assumed to be continuously differentiable, implementable and extendable by continuity to $\overline{\Omega}$.[7] Let $x_0(\alpha)$ be the outside option implementing $s_0(\alpha)$ at price $p(\alpha)$ for type α, so that $v(\alpha, x_0(\alpha), p(\alpha)) = s_0(\alpha)$.

The firm looks for a selling mechanism that maximizes its profits. The Taxation Principle (Rochet [22]) implies that one can, without loss of generality, assume that the monopolist simply announces a non-linear tariff $t : X \to \mathbb{R}$.

The above considerations can be summarized by the following model. The firm selects a function $t : X \to \mathbb{R}$ to solve

$$\max \int_\Omega (t(x(\alpha)) - c(x(\alpha)))f(\alpha)d\alpha, \tag{1}$$

where $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_{x \in X} v(\alpha, x, t(x)) & \text{if } \max_{x \in X} v(\alpha, x, t(x)) \geq s_0(\alpha) \\ x(\alpha) = x_0(\alpha) & \text{otherwise.} \end{cases} \tag{2}$$

Define the net utility as the unique function $u(\alpha, x)$ that solves

$$s_0(\alpha) = v(\alpha, x, u(\alpha, x)). \tag{3}$$

[5]It is convenient for our purposes to use some (basic) concepts and results from geometric measure theory, and we refer the reader to Evans and Gariepy [12] and Chlebik [10] for more information. For the reader who is not familiar with geometric measure theory, it suffices to keep in mind that the type spaces that we consider are allowed to be more general than the typical type spaces in the literature. We also refer the reader for the examples in Section 3 for some concrete geometric and economic intuition.

[6]Recall that the characteristic function of Ω is given by $\chi_\Omega(x) = 1$ if $x \in \Omega$ and $\chi_\Omega(x) = 0$ if $x \notin \Omega$. Hence, a set of finite perimeter can have holes and a rough (i.e., not Lipschitz) boundary, provided that the latter is not "too wiggly", in the sense that the variation of χ_Ω has to remain bounded.

[7]For conditions of implementability of a surplus function see Basov [5].

The economic meaning of $u(\alpha, x)$ is the maximal amount type α is willing to pay for the bundle x. Given our assumptions on v_t, u is differentiable by the Implicit Function Theorem. Note that the optimal tariff paid by type α satisfies

$$t(x(\alpha)) \leq u(\alpha, x(\alpha)). \tag{4}$$

Let $s(\alpha)$ denote the surplus obtained by type α:

$$s(\alpha) = \begin{cases} \max_{x \in X} v(\alpha, x, t(x)) - s_0(\alpha) & \text{if } \max_{x \in X} v(\alpha, x, t(x)) \geq s_0(\alpha) \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Accordingly, we have the envelope condition

$$\nabla s(\alpha) = \nabla_\alpha v(\alpha, x(\alpha), t(x(\alpha))) - \nabla s_0(\alpha)$$

which holds for almost every α with $x(\alpha) \neq x_0(\alpha)$. From (3) we have

$$\nabla s_0(\alpha) = \nabla_\alpha v(\alpha, x(\alpha), u(\alpha, x(\alpha))) + v_t(\alpha, x(\alpha), u(\alpha, x(\alpha)))\nabla_\alpha u(\alpha, x(\alpha)),$$

so the envelope condition plus our assumptions on v_t and $v_{i,t}$ yield

$$\lambda(\alpha)\nabla s(\alpha) \geq \nabla_\alpha u(\alpha, x(\alpha)) \tag{6}$$

for almost every α with $x(\alpha) \neq x_0(\alpha)$, where $\lambda(\alpha) = |v_t(\alpha, x(\alpha), u(\alpha, x(\alpha)))|^{-1}$ is positive and bounded away from zero by assumption.

We are interested in the the set of **excluded consumers**, given by

$$\{\alpha \in \Omega : x(\alpha) = x_0(\alpha)\},$$

that is, the set of types that optimally choose to not participate.

Assumption 1. *u is strictly increasing in α for each $x \in X$.*

For $a, b \in \mathbb{R}^m$ let $a \cdot b$ denote the inner product of a and b.

Assumption 2. *There exists $K > 0$ such that $u(\alpha, x) \leq K\alpha \cdot \nabla_\alpha u(\alpha, x)$ for every $(\alpha, x) \in \overline{\Omega} \times \overline{X}$.*

Assumptions 1 and 2 are regularity conditions, requiring that the net utility be increasing in α and bounded. Note that v is allowed to be decreasing in α for each (x, t), as long as Assumptions 1 and 2 are satisfied.

For any Lebesgue measurable set $E \subset \mathbb{R}^m$ let $\mathcal{L}^m(E)$ denote its Lebesgue measure and $\mathcal{H}^s(E)$ denote its s-dimensional Hausdorff measure. For $s = m$, the Hausdorff measure of a Borel set coincides with the Lebesgue measure, while for $s < m$ it generalizes the notion of the surface area.[8]

Let $\partial_e\Omega$ denote the measure-theoretic boundary of Ω.[9] Because Ω has locally finite perimeter, the measure-theoretic boundary can be decomposed into countably many C^1 pieces and a residual set with measure zero. That is,

$$\partial_e\Omega = \bigcup_{i=1}^{\infty} K_i \cup N,$$

where K_i is a compact subset of a C^1-hyper-surface S_i, for $i \geq 1$, and $\mathcal{H}^{m-1}(N) = 0$. Accordingly, let us write

$$K_i = \{\alpha \in \overline{\Omega} : g_i(\alpha) = 0\}$$

for $i \geq 1$, with $g_i : \mathbb{R}^m \to \mathbb{R}$ of class C^1.

Assumption 3. *For all $\alpha \in \mathbb{R}^m$ and $i \geq 1$, we have*

$$\nabla_\alpha g_i(\alpha) \in \mathbb{R}^m \backslash (\mathbb{R}^m_+ \cup \mathbb{R}^m_-).$$

Assumption 3 restricts Ω to be of a particular class of type spaces, which includes the "rightward-slanted diamond" type spaces mentioned in the Introduction (see also Figure 2 in Section 3.) Together with Assumption 1, it implies that the boundary of Ω is never parallel to the iso-surplus hyper-surfaces. As we argue in Section 3 below, this is the key geometric requirement for obtaining exclusion.

Our main results come next. They will be stated without reference to the well-known sufficient conditions for implementability of the surplus function s in order to focus on the conditions that highlight the nature of the contribution being made.

[8]For a definition of the Hausdorff measure, see Chlebik [10].

[9]The measure-theoretic boundary of a set is contained in the topological boundary of the set. It consists of the points that are neither in the measure-theoretic interior nor in the measure-theoretic exterior of the set, which are supersets of their topological counterparts.

Theorem 1. *Consider the problem (1)-(2), and assume that it has a finite solution yielding an allocation $x(\alpha)$ and surplus $s(\alpha)$ which are continuous. Then, under Assumptions 1, 2 and 3, the set of excluded consumers at the solution has positive measure.*

That is, if the problem has a continuous solution, Assumptions 1, 2 and 3 ensure that there will be exclusion.

Remark 1. *The conditions in Theorem 1 are in general more permissive than the conditions in the literature, including those used by Armstrong [1]. In particular, the literature focuses on the quasilinear case, where the net utility is simply $u(\alpha, x) = \nu(\alpha, x) - s_0(\alpha) = \nu(\alpha, x)$, as $s_0(\alpha)$ is usually assumed to be equal to zero for every α. Armstrong [1] assumes that $\nu(\cdot, x)$ is homogeneous of degree 1 and strictly convex, which are special cases of Assumptions 1 and 2. As for the type space Ω, the assumptions are non-nested: Assumption 3 allows for non-convex, general type spaces, but with a particular boundary structure, whereas Armstrong's assumption that Ω is strictly convex allows for type spaces with boundary structures violating Assumption 3.*

Consider now an underlying *set of all type spaces.* It is given by $\{\Omega_\beta : \beta \in \mathcal{B}\}$, where $\mathcal{B}$ is an index set. For each $\beta \in \mathcal{B}$, Ω_β is an open set with locally finite perimeter in some open set U_β and its boundary structure is given by

$$\partial_e \Omega_\beta = \bigcup_{i=1}^{\infty} K_{i,\beta} \cup N_\beta,$$

where

$$K_{i,\beta} = \{\alpha \in \overline{\Omega}_\beta : g_i(\alpha, \beta) = 0\}$$

for $i \geq 1$, with $g_i : \mathbb{R}^m \times \mathcal{B} \to \mathbb{R}$ of class C^1, and N_β is a set of $\mathcal{H}^{m-1}$-measure zero. We make the following assumption about $\{\Omega_\beta : \beta \in \mathcal{B}\}$:

Assumption 4. *(i) $\mathcal{B}$ is a finite dimensional open manifold of class C^1; (ii) the family $\{g_i\}_{i\geq 1}$ is compact in the C^1 topology;*[10] *(iii) there exists $\hat{\beta} \in \mathcal{B}$ such that*

$$\nabla_\alpha g_i(\alpha, \hat{\beta}) \in \mathbb{R}^m \backslash (\mathbb{R}^m_+ \cup \mathbb{R}^m_-)$$

for all $\alpha \in \mathbb{R}^m$ and all $i = 1, ..., k$.

[10] See Hirsh [15] for a definition of the C^1 topology. Intuitively, proximity in the C^1 topology means proximity of values and first derivatives.

That is, the parameters $\mathcal{B}$ determine the underlying set of type spaces that we consider. Requirements (ii) and (iii) are mild technical requirements ensuring that we can apply transversality ideas to establish that exclusion is generic: the compactness requirement in (ii) is a regularity condition that is satisfied if we restrict to boundary structures formed of only finitely many pieces, as it is the case in all applications we are aware of; (iii) simply requires that at least one type space of the kind identified in Assumption 3 be included as a member of the allowed types spaces. A seemingly more important requirement is the finite dimensionality of $\mathcal{B}$ in requirement (i). But this is just for a cleaner presentation of our ideas. In Lemma 6 in the Appendix, we make use of the standard Transversatiliy Theorem, which is valid in a finite dimensional environment. It is well known that there exist general versions of the Transversality Theorem that allow for infinite dimensions.[11] One can generalize Assumption 4 allowing for an infinite dimensional $\mathcal{B}$ and adapt Lemma 6 with a more powerful Transversality Theorem. We leave this task to the interested reader. Let $\varphi : \mathcal{B} \rightrightarrows \mathbb{R}^m$ be given by $\varphi(\beta) = \Omega_\beta$.

Theorem 2. *Consider the problem (1)-(2) for each $\beta \in \mathcal{B}$, and assume that s_0 and ∇s_0 are continuous at each (α, β) in the closure of the graph of φ. Assume that the problem has a finite solution yielding an allocation $x(\alpha; \beta)$ and surplus $s(\alpha; \beta)$ which are continuous at each (α, β) in the closure of the graph of φ. Then, under Assumptions 1, 2 and 4, for a generic model β (that is, for all β in an open and dense subset of $\mathcal{B}$), the set of excluded consumers at the solution has positive measure.*

That is, if the problem has a continuous solution, Assumptions 1, 2 and 4 ensure that, generically, there will be exclusion.

Remark 2. *The conditions in Theorem 2 are strictly more general than the conditions in the literature. In particular, the inclusion of a rightward-slanted diamond kind of type space as one of the allowed type spaces (part (iii) of Assumption 4) is without loss of generality. It is only when combined with parts (i) and (ii) of Assumption 4 that part (iii) has a (mild) bite.*

[11] See Golubitsky and Guillemin [14] for the relevant concepts in the theory of transversality. The version we use can be found in Mas-Colell, Whinston and Green [20], Theorem M.E.2.

3 Discussion

We now provide some geometric and economic intuition behind our results. Let us start with the setting in Armstrong [1], illustrated in Figure 1 below. The type space Ω is a strictly convex set in $\mathbb{R}^m$, for instance, a ball. At the solution, we compute the surplus function as in (5) above, and consider the iso-surplus hyper-surfaces $s_\varepsilon^{-1} = \{\alpha \in \Omega : s(\alpha) = \varepsilon\}$, for some $\varepsilon \in \mathbb{R}_+$. Under Armstrong's [1] assumptions, the surplus function is convex, so, in the two-dimensional case illustrated in Figure 1, s_ε^{-1} is a concave curve. The set of types below s_ε^{-1} are the types enjoying less than ε surplus. If the set of excluded types is of measure zero, then s_0^{-1} will (in the case depicted) be tangent to the boundary $\partial\Omega$ of Ω. Now consider increasing the tariff by $\varepsilon > 0$ so that types below s_ε^{-1} find it optimal to not participate. We argue that, for the monopolist, the loss in profit from losing these types is more than compensated by the gain in profit obtained by selling at a uniformly higher tariff for every type above s_ε^{-1}. In fact, the gain in profit is of the order $O(\varepsilon)$. The loss is proportional to the region below s_ε^{-1}, which, for small ε, is approximately a simplex with height

$$h = \frac{\varepsilon}{||\nabla s(\alpha)||},$$

for some α in the region, and base given by the $m-1$ dimensional measure $\mathcal{H}^{m-1}$ of the part of the boundary $\partial\Omega$ below s_ε^{-1}. This measure is in turn proportional to a^{m-1}, with

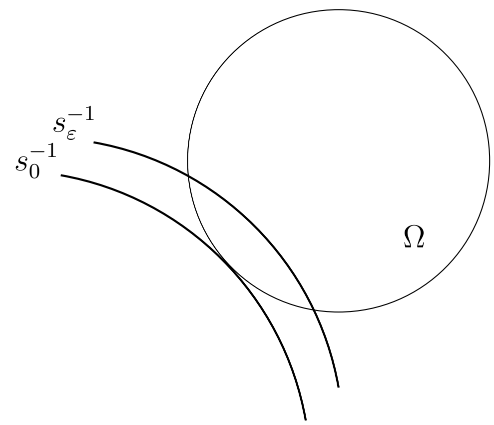
$$a = \varepsilon \cot(\gamma)$$

where γ is the angle between the normal vectors to s_ε^{-1} and $\partial\Omega$. That is, the loss is proportional to $a^{m-1}h$ which is of the order $O(\varepsilon^m)$. Hence the loss is infinitely smaller than the gain, which then means that the monopolist was not optimizing by not excluding a positive measure of types.

Observe that the logic above breaks down when s_0^{-1} is parallel to $\partial\Omega$, because then $\gamma = 0$ and $\cot(\gamma) = \infty$. In fact, when s_0^{-1} is parallel to $\partial\Omega$, the types are essentially one-dimensional (by identifying the type with the distance from the boundary to the corresponding iso-surplus hyper-surface) and exclusion need not be optimal.[12]

[12]See Basov [6] for conditions ensuring that multi-dimensional types can be summarized by a one-dimensional variable.

Figure 1: Armstrong's Setting



Also note that, when s_0^{-1} is parallel to $\partial\Omega$, the measure of the intersection of s_0^{-1} and $\partial\Omega$ is not $\mathcal{H}^{m-1}$-null, as opposed to the case illustrated in Figure 1. In fact, the logic described above carries through under much more general conditions, as illustrated in Figure 2, where the type space is a rightward-slanted diamond and the iso-surplus hyper-surfaces are more general (that is, are neither convex nor concave curves in the two-dimensional case illustrated.) Because the intersection of any iso-surplus hyper-surface and the boundary of Ω is at most a two-point set (in the m-dimensional case depicted, with $m = 2$), it is $\mathcal{H}^{m-1}$-null. Again, if there set of excluded types is of measure zero, then s_0^{-1} will intersect $\partial\Omega$ only at one point, and similarly to the argument above, increasing the tariff by $\varepsilon > 0$ will generate a loss of order $O(\varepsilon^m)$ and a gain of order $O(\varepsilon)$. In fact, the loss is the area below s_ε^{-1}, which is again approximately a simplex with height proportional to ε and base of order $O(\varepsilon^{m-1})$, so it is of order $O(\varepsilon^m)$.

Figure 2: The Rightward-Slanted Diamond

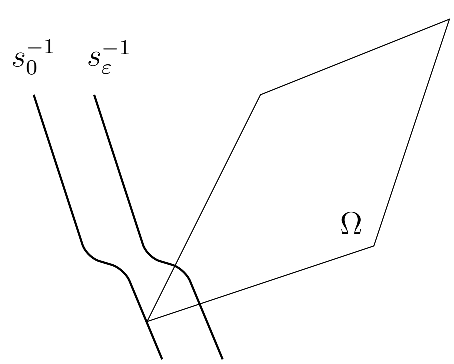
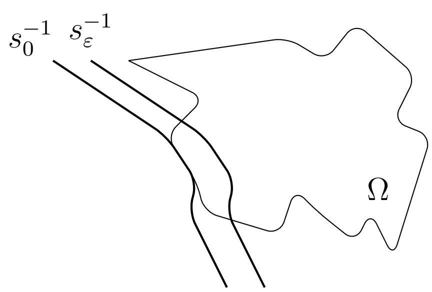


Figure 2 illustrates Theorem 1: as long as the boundary structure of Ω is such that none of the normal vectors of the components of the boundary is strictly positive or strictly negative, the intersection of an iso-surplus hyper-surface and $\partial\Omega$ will be $\mathcal{H}^{m-1}$-null, and it will be optimal to exclude a set of positive measure of types.

In general, the boundary structure of Ω may not satisfy Assumption 3, as it does in Figure 2. For instance, consider the situation in Figure 3. The iso-surplus hyper-surfaces are again general as in Figure 2, the type space is not convex and the boundary structure fails Assumption 3. In the case illustrated, the intersection of s_0^{-1} and $\partial\Omega$ is not $\mathcal{H}^{m-1}$-null. By increasing the tariff by $\varepsilon > 0$, the measure of types that will optimally not participate is not of order $O(\varepsilon^m)$ anymore. It is again given by the types below s_ε^{-1}, a set whose measure is order $O(\varepsilon)$, as it is approximately a hyper-cube, with base of positive $\mathcal{H}^{m-1}$ measure and height of ε.

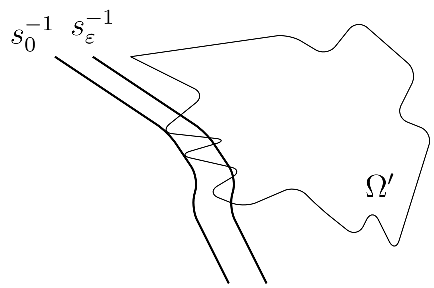
Figure 3: A More General Setting



Now consider a model Ω' that is close to the model Ω in Figure 3, as illustrated in Figure 4.[13] Now we again have a $\mathcal{H}^{m-1}$-null intersection of s_0^{-1} and $\partial\Omega$, and the same argument (now with three simplices of order $O(\varepsilon^m)$) establishes that the cost of increasing the tariff by ε is of order $O(\varepsilon^m)$ and the gain is of order $O(\varepsilon)$ if there was no exclusion. That is, Figure 4 illustrates Theorem 2: generically, the intersection of the boundary structure of a type space and the hyper-surface s_0^{-1} will be $\mathcal{H}^{m-1}$-null, so for a generic type space the monopolist will do better by excluding a positive measure of types.

[13]The depicted iso-surplus curves need not coincide with the ones in Figure 3, as the changed type space will affect the choices. But, by continuity, the new curves will be close to the old ones, so the illustration in Figure 4 is accurate. We should also note that both Ω and Ω' are sets of locally finite perimeter satisfying part (ii) of Assumption 4.

Figure 4: A Type Space Close-by



4 Examples

Before proceeding to the proofs, let us present some implications for economic applications, and also some examples to illustrate the nature of the results.

4.1 Applications

We first present a completion of the analysis in Armstrong [2] and then an application to involuntary unemployment.

Example 1. Armstrong [2] reviews Lewis and Sappington's [19] analysis of optimal regulation of a monopolist with two-dimensional private information. A single product monopolist faces a stochastic demand function given by $q\,(p) = a+\theta-p$, where p is the price of the product, $a > 0$ is a fixed parameter, and θ is a stochastic component to demand, taking values in an interval $[\underline{\theta}, \overline{\theta}] \subset \mathbb{R}_+$. The firm's cost is represented by the function $C\,(q) = (c_0 + c)\,q + F$, where q is the quantity produced, $c_0 > 0$ and $F > 0$ are fixed parameters and c is a stochastic component to the cost, taking values in an interval $[\underline{c}, \overline{c}] \subset \mathbb{R}_+$. The firm observes both the demand and the cost functions, while the regulator only knows that $\alpha = (\theta, c)$ is distributed according to a strictly positive continuous density function $f\,(\theta, c)$ on the rectangle $\Omega = \left[\underline{\theta}, \overline{\theta}\right] \times [\underline{c}, \overline{c}]$. For feasibility, we assume that $a + \underline{\theta} > c_0 + \overline{c}$, i.e., the highest demand exceeds marginal costs, for all possible realizations of the stochastic components of demand and cost.

The regulator wants to maximize social welfare and presents to the monopolist a menu of contracts $\{p, t(p)\}$, where a contract has the monopolist sell the product at unit price p and pay a tax $t(p)$ to the regulator. Negative values of t represent subsidies. Social welfare is given by the sum of the consumers' surplus $\frac{1}{2}(a+\theta-p)^2$ and profits $pq(p)-c(q)-t(p)$. Let us employ a change of variables to represent the problem of the regulator in the form of the problem in Section 2. Set $x=p$, and $X=(c_0+\overline{c}, a+\underline{\theta})$ as the set of feasible prices; let

$$\nu(\alpha,x)=(a+\theta-x)(x-c_0-c)-F$$

be the profit before the tax, and let

$$c(x)=-\frac{1}{2}(a+\theta-x)^2-(a+\theta-x)(x-c_0-c)+F+t(x)$$

be the negative of welfare net of taxes. Finally, let $x_0(\alpha)$ be the outside option yielding $s_0(\alpha)=0$.

The problem of the regulator is to select a tax schedule $t: X\to\mathbb{R}$ to solve

$$\max\int_\Omega (t(x(\alpha))-c(x(\alpha)))f(\alpha)d\alpha,$$

where $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha)\in\arg\max_{x\in X}\nu(\alpha,x)-t(x) & \text{if } \max_{x\in X}\nu(\alpha,x)-t(x)\geq 0 \\ x(\alpha)=x_0(\alpha) & \text{otherwise.}\end{cases}$$

The choice of $x(\alpha)$ by the monopolist depends on whether she can derive nonnegative returns when producing. If that is not possible, she will choose the outside option $x_0(\alpha)$.

Lewis and Sappington [19] assume that the parameter a can be chosen sufficiently large relative to parameters F and c_0 so that a firm will always find it in its interest to produce, even for the very small values of θ. However, Armstrong [2] shows that such a hypothesis cannot be made when Ω is the square $[0,1]\times[0,1]$. Furthermore, when Ω is a strictly convex subset of this square, Armstrong [2] uses the optimality of exclusion theorem in Armstrong [1] to show that some firms will necessarily shut down under the optimal regulatory policy. Armstrong [2] then adds "... I believe that the condition that the support be convex is *strongly* sufficient and that it will be the usual case that exclusion is optimal, even if a is much larger than the maximum possible marginal cost." That insight could not be pursued further due to a lack of a more general result, and Armstrong [2] switched to a discrete-type model in order to check the robustness of the main conclusions in Lewis and Sappington [19].

It is clear that the regulator's problem is essentially the standard problem solved in Section 2 above. Note that, given the specification of X above, Assumptions 1 and 2 are met. Assumption 4 is met as soon as we include the rightward-slanted diamonds as allowed type spaces, as for instance the following slight change of the square: $\text{co}\{(0,0),(\varepsilon,1),(1+\varepsilon,1+\varepsilon),(1,\varepsilon)\}$, for $\varepsilon > 0$, where "co" stands for convex hull. All the hypothesis of Theorem 2 are satisfied, so we may conclude that a set of positive measure of firms will generically be excluded from the regulated market. Armstrong's [2] conjecture is therefore confirmed, generically.

We can actually say more about the matter for the given type space. The gradient of the surplus $s(\alpha)$ is $\nabla s(\alpha) = (x(\alpha) - c_0 - c, a - \theta - x(\alpha))$, and the boundary of the square is described by g_i, $i = 1, ..., 4$ with $\nabla g_i(\alpha) = (1, 0)$ for $i = 1, 3$ and $\nabla g_i(\alpha) = (0, 1)$ for $i = 2, 4$. Hence the rank of

$$\begin{pmatrix} \nabla s(\alpha) \\ \nabla g_i(\alpha) \end{pmatrix}$$

is equal to two for $i = 1, ..., 4$, meaning that the iso-surplus curves are never parallel to the boundaries of the square (as $x(\alpha) \in X$ for every α.) Hence, even though the square $[0,1] \times [0,1]$ violates Assumption 3, the intersection of the iso-surplus curves and the boundary of the type space will be $\mathcal{H}^1$-null, as long as the iso-surplus curves do not become asymptotically vertical or horizontal as x gets close to the boundaries of X. If the set of allowed prices is chosen to be an interval strictly inside of $(c_0 + 1, a) = (c_0 + \bar{c}, a + \underline{\theta})$, then the gradient $\nabla s(\alpha)$ will be bounded away from $(1, 0)$ and $(0, 1)$, and the iso-surplus curves will never be either almost vertical or almost horizontal. As a result, the intersection of s_0^{-1} (recall the arguments and notation from Section 3) and the boundary of the square will be $\mathcal{H}^1$-null, so an ε increase in the subsidy schedule from a situation with no exclusion will lose an order $O(\varepsilon^2)$ of firms and generate a gain of order $O(\varepsilon)$. Again, this establishes that the regulator will exclude a positive mass of firms. Importantly, observe that restricting X to be strictly inside of $(c_0 + 1, a)$ goes in line with Lewis and Sappington's [19] idea that a firm will always find in its interest to produce. In fact, such a restriction can be viewed as avoiding *a priori* exclusion, for prices below cost or above the highest demand for a given type of the firm will certainly exclude the firm. Yet, due to the geometry of exclusion, not excluding firms *a priori* necessarily leads to exclusion of a positive mass of firms once the optimal contract is implemented. □

Example 2. Another interesting application concerns the emergence of involuntary unemployment as a consequence of adverse selection. Consider a firm in an industry that produces n goods captured by a vector $x \in X \subset \mathbb{R}^n_+$. The firm hires workers to produce these goods. A worker is characterized by the cost she bears in order to produce goods $x \in \mathbb{R}^n_+$, which is given by the effort cost function $e(\alpha, x)$. The parameter $\alpha \in \Omega \subset \mathbb{R}^m$ is the worker's unobservable type distributed on an open, bounded set $\Omega \subset \mathbb{R}^m$ according to a strictly positive, continuous density function f.

Therefore, if a worker of type α is hired to produce output x and receives wage $\omega(x)$, her utility is $\omega(x) - e(\alpha, x)$, where $e(\alpha, \cdot)$ is cost of effort, which depends on the type of the worker. If the worker is not hired by the firm, she will receive a net utility $s_0(\alpha)$, either by working for a different firm, or by receiving unemployment compensation.

Let $p : X \to \mathbb{R}_+$ denote the revenue function that the firm faces, and assume it is a concave function.

The firm's problem is to select a wage schedule $\omega : R_+^n \to \mathbb{R}$ to solve

$$\max \int_\Omega [p(x(\alpha)) - \omega(x(\alpha))] f(\alpha) d\alpha,$$

where $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_{x \in X} \omega(x) - e(\alpha, x) & \text{if } \max_{x \in X} \omega(x) - e(\alpha, x) \geq s_0(x) \\ x(\alpha) = 0 & \text{otherwise.} \end{cases}$$

Consider the following change in variables: $t(x) = -\omega(x)$, $\nu(\alpha, x) = -e(\alpha, x)$, $c(x) = -p(x)$. Then the firm's problem can be rewritten as

$$\max \int_\Omega (t(x(\alpha)) - c(x(\alpha))) f(\alpha) d\alpha,$$

where $x(\alpha)$ satisfies

$$\begin{cases} x(\alpha) \in \arg\max_{x \in X} \nu(\alpha, x) - t(x) & \text{if } \max_{x \in X}(\nu(\alpha, x) - t(x)) \geq s_0(x) \\ x(\alpha) = 0 & \text{otherwise.} \end{cases}$$

Therefore, the arguments for the monopolist screening problem can be extended to the hiring decision of the firm. In particular, the firm will generically find it optimal not to hire a set of positive measure. If the firm is a monopsonist in the sense that agents can work only at that firm, then our main result provides a rationale for involuntary unemployment. □

4.2 No-Exclusion is Knife-Edge

The next three examples illustrate why the main result is only about the generic case. There are non-generic cases where no consumer is excluded.

Example 3. Consider a problem that yields an excluded set Ω_0 with positive measure, and modify the problem by considering only the types in $\Omega \backslash \Omega'$, where $\Omega_0 \subset \Omega'$. Would the modified problem have no exclusion? The answer is yes if $\Omega' = \Omega_0$, but the answer is no if Ω' is a generic superset of Ω_0.[14] That is, the shape of Ω' has to bear a tight relation to the shape of Ω_0 for no exclusion to hold in the model $\Omega \backslash \Omega'$. And of course, even if this tight relation holds, a slight change in the boundary structure of Ω would suffice for us to have exclusion in the modified model. □

Example 4. Consider the example presented by Rochet and Stole [24]. The quasilinear utility of agent of type α is given by

$$\nu(\alpha, x) = (\alpha_1 + \alpha_2)x$$

and Ω is a rectangle with sides parallel to the 45 degrees and -45 degrees lines. Rochet and Stole argued that one can shift the rectangle sufficiently far to the right to have an empty exclusion region. Their result is driven by the fact that they allow only very special collections of type spaces, rectangles with parallel sides. Formally, the type spaces allowed violate Assumptions 3 and 4(iii) as the boundary structure is given by $\nabla_\alpha g_i(\alpha, \beta) = (1, 1) \in \mathbb{R}^2_+$ for $i = 1, 3$ and $\nabla_\alpha g_i(\alpha, \beta) = (-1, 1)$ for $i = 2, 4$ (where i labels, in a clockwise fashion starting from the southwest side, the sides of a rectangle with sides parallel to the 45 and -45 degrees lines, and β represents right-shits of such rectangles). In fact, note that (using $u(\alpha, x) = \nu(\alpha, x)$ because $s_0(\alpha) = 0$)

$$\begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta) \end{pmatrix} = \begin{pmatrix} x & x \\ 1 & 1 \end{pmatrix}$$

for $i = 1, 3$, so it is a matrix of rank equal to 1, which means that the iso-surplus curves (more precisely, lines) are parallel to the downward sloping sides of the type space.[15] As we saw in Section 3, in such cases the cost-benefit analysis does not lead us to conclude that exclusion is optimal (in fact, as argued there as well, this is a case where types can be summarized by an one-dimensional variable, so the particularities of the multi-dimensional setting are lost.)

Observe that a very small change in the type set changes that result. Consider, for example, a slightly perturbed type space, with $\nabla_\alpha g_i(\alpha, \beta_0) = (1, 1 + \varepsilon)$, for $i = 1, 3$, where

[14] We are grateful to an anonymous referee for this observation.

[15] Observe that here $\nabla_\alpha u(\alpha, x(\alpha))$ is the gradient of the surplus function, $\nabla s(\alpha)$.

$\varepsilon > 0$ is a small positive real number. Then, for all $x \neq 0$ and $i = 1, ..., 4$

$$\text{rank} \begin{pmatrix} \nabla_\alpha u(\alpha, x) \\ \nabla_\alpha g_i(\alpha, \beta_0) \end{pmatrix} = 2,$$

so that the iso-surplus lines will cut transversally the boundary of the perturbed type space. As a result, the intersection of these two will be $\mathcal{H}^1$-null and, using the argument in Section 3, we obtain that exclusion is optimal. In words, Rochet and Stole [24] is a knife-edge case of no exclusion: slight perturbations of the boundary structure of their type spaces suffice to generate exclusion. □

Example 5. Consider the model of Armstrong and Vickers [3], which allows for multi-dimensional vertical types. In this kind of models, the type consists of a vector of vertical characteristics, $\alpha \in \Omega \subset \mathbb{R}^m$ as in Section 2, and a parameter $\gamma \in [0, 1]$ capturing horizontal preferences. The type space is given by the Cartesian product $\Omega \times [0, 1]$ and γ is assumed to be distributed independently of α. The utility of a consumer is given by

$$\nu(\alpha, x; \gamma) = \upsilon(\alpha, x) - r\gamma,$$

where r is a commonly known parameter, and $x \in X \subset \mathbb{R}^n_+$ is the vector of goods. Let $\upsilon(\alpha, 0) = 0$ so that the iso-surplus hyper-surface corresponding to $x = 0$ is $r\gamma =$ constant, which is parallel to the vertical boundary of the type space, $\gamma = 0$. Therefore, in such a model there is the possibility of no exclusion. The model was also investigated in an oligopolistic setting, where r was interpreted as a transportation cost for the Hotelling model. Conditions for no exclusion under different assumptions on the dimensionality of α and the monopolist's risk preferences were obtained by Armstrong and Vickers [3], Rochet and Stole [23], and Basov and Yin [7]. Nevertheless, these are all knife-edge cases. In fact, let us parametrize the boundary of the set Ω so that it is described by the equation

$$g_0(\alpha) = 0$$

and embed our problem into a family of problems, for which the boundary of the type space is described by the equation

$$g(\alpha, \gamma; \beta) = 0,$$

where $g(\cdot, \beta) : \Omega \times [0, 1] \to \mathbb{R}$ is a C^1 function with

$$g(\alpha, \gamma; 0) = g_0(\alpha)(g_0(\alpha) - b)\gamma(\gamma - 1)$$

for some constant b. Note that this establishes that, for $\beta = 0$ the type space is the cylinder over the set Ω considered by Armstrong and Vickers [3] for which the exclusion region is empty.[16] But under Assumption 4, among the possible underlying type spaces there will

[16]The level set $g(\alpha, \gamma, 0) = 0$ of the displayed equation is exactly the said cylinder.

be at least one with boundaries with $\mathcal{H}^{m-1}$-null intersections with the iso-surplus curves. For this type space the exclusion region will be non-empty. And a standard transversality argument establishes that the same is true for almost all β. □

4.3 An Illustration of the Assumptions

We conclude this list of examples with an illustration of Assumptions 1 – 4.

Example 6. Consider a consumer who lives for two periods. Her wealth in the first period is w and in the second period her wealth can take two values, w_H or w_L. Let p be the probability that $w = w_H$, and let $\delta \in (0,1)$ be the discount factor, so that the private information of the consumer is characterized by a two-dimensional vector $\alpha = (1-p, 1-\delta)$. The consumer's preferences are given by:

$$V(c_1, c_2) = \upsilon(c_1) + \delta E\upsilon(c_2)$$

where c_1 and c_2 are the consumption levels in periods 1 and 2 respectively, and $\upsilon(\cdot)$ is increasing with its derivative υ' bounded away from zero. Assume that wealth is not storable between periods. Instead, the consumer can borrow x from a bank in period 1, and repay t in period 2 if $w = w_H$, and to defaut if $w = w_L$ in period 2. If the consumer does not borrow, her expected utility will be

$$s_0(\alpha) = \upsilon(w) + \delta(p\upsilon(w_H) + (1-p)\upsilon(w_L)),$$

which is the type dependent outside option. If she borrows x and repays t, the expected utility will be

$$v(\alpha, x, t) = \upsilon(w + x) + \delta(p\upsilon(w_H - t) + (1-p)\upsilon(w_L)),$$

which is strictly increasing in x and strictly decreasing in t. Let $\Omega_1 = (0,1)^2$ be the type space, with boundary captured by $g_i(\alpha, \beta_1)$, $i = 1, ..., 4$, with $\nabla_\alpha g_i(\alpha, \beta_1) = (0, 1)$ for $i = 1, 2$ and $\nabla_\alpha g_i(\alpha, \beta_1) = (1, 0)$ for $i = 3, 4$. Assumption 3 is violated by Ω_1. But let Ω_2 be another type space, included in the underlying space of type spaces, with boundary given by $g_i(\alpha, \beta_2)$, $i = 1, ..., 4$, with $\nabla_\alpha g_i(\alpha, \beta_2) = (-\varepsilon, 1)$ for $i = 1, 2$ and $\nabla_\alpha g_i(\alpha, \beta_2) = (1, -\varepsilon)$ for $i = 3, 4$, for some $\varepsilon > 0$. Assumption 4(iii) is thus met.

As

$$\nabla_\alpha u(\alpha, x) = \left(\frac{\Delta\upsilon}{p\upsilon'}, \frac{\Delta\upsilon}{\delta\upsilon'}\right),$$

where $\Delta\upsilon = \upsilon(w_H) - \upsilon(w_H - u(\alpha, x)) > 0$, Assumptions 1 and 2 are met as well. □

Example 6 above is a natural setting to discuss unavailability of credit to some individuals, which is important to justify monetary equilibria in the search theoretic models of money.[17]

5 Conclusion

We showed that, in general, exclusion is optimal in multi-dimensional monopolistic screening problems. In particular, we provided novel sufficient conditions for exclusion and showed that, under a sufficiently rich underlying set of type spaces, if one such type space satisfies the sufficient conditions for exclusion, then exclusion obtains for a generic type space. The results rely on the straightforward geometric idea that if the iso-surplus hyper-surfaces cut the boundary of the type space transversally, then the monopolist cannot be optimizing by including a set of full measure of types: a slight uniform increase in the tariff will certainly increase profits, as only a negligible fraction of types will drop out.

A Proofs

We divide the proofs into several intermediate steps. Let $\mathcal{K}(\mathbb{R}^m)$ be the hyperspace of compact sets in $\mathbb{R}^m$, endowed with the topology induced by the Hausdorff distance d_H, given by

$$d_H(E, F) = \inf\{\varepsilon > 0 : E \subset F^\varepsilon, F \subset E^\varepsilon\},$$

where

$$E^\varepsilon = \bigcup_{\alpha \in E} B(\alpha, \varepsilon)$$

and $B(\alpha, \varepsilon)$ is the open ball centered at α and with radius $\varepsilon > 0$. Because

$$\lim_{\varepsilon \to 0+} \mathcal{L}^m(E^\varepsilon) = \mathcal{L}^m(E), \quad \lim_{\varepsilon \to 0+} \mathcal{H}^s(E^\varepsilon) = \mathcal{H}^s(E)$$

for all $s \geq 0$, both $\mathcal{L}^m$ and $\mathcal{H}^s$ are upper semicontinuous functions in $\mathcal{K}(\mathbb{R}^m)$ (Beer [8]).

Lemma 1. *Let $E \in \mathcal{K}(\mathbb{R}^m)$ be such that $\mathcal{L}^m(E) = \mathcal{H}^s(E) = 0$, for some $s \geq 0$, and let $(E_k)_{k \geq 1}$ be a sequence in $\mathcal{K}(\mathbb{R}^m)$ such that $E_k \to E$. Then $\mathcal{L}^m(E_k) \to 0$ and $\mathcal{H}^s(E_k) \to 0$.*

[17] See, for example, Lagos and Wright [18].

Proof. Because $\mathcal{L}^m$ is a non-negative upper semicontinuous set function, we have

$$\lim \inf_{E_k \to E} \mathcal{L}^m(E_k) \geq 0 = \mathcal{L}^m(E) \geq \lim \sup_{E_k \to E} \mathcal{L}^m(E_k),$$

so $\mathcal{L}^m(E_k) \to 0$, and analogously for $\mathcal{H}^s$. □

That is, Lemma 1 establishes continuity of Lebesgue and Hausdorff measures at zero.

Let us write $\Omega_0 = \{\alpha \in \Omega : s(\alpha) = 0\}$, where $s(\alpha)$ is the surplus obtained by type α. Extending s by continuity to $\partial\Omega$, let $\overline{\Omega}_0 = \{\alpha \in \overline{\Omega} : s(\alpha) = 0\}$. We also extend x by continuity to $\partial\Omega$.

Lemma 2. *Under Assumption 1, $\mathcal{L}^m(\overline{\Omega}_0) = 0$ implies $\overline{\Omega}_0 \subset \partial\Omega$.*

Proof. If $\overline{\Omega}_0 \not\subseteq \partial\Omega$, there is $\alpha \in \Omega_0$ and $\varepsilon > 0$ with $B(\alpha, \varepsilon) \subset \Omega$. Then

$$\mathcal{L}^m(\{\hat{\alpha} \in \overline{\Omega} : \hat{\alpha} \leq \alpha\} \cap B(\alpha, \varepsilon)) > 0.$$

Because of Assumption 1, we cannot have $s(\hat{\alpha}) > 0$ for any $\hat{\alpha} \leq \alpha$, for otherwise $s(\alpha) > 0$ as well. So

$$\{\hat{\alpha} \in \overline{\Omega} : \hat{\alpha} \leq \alpha\} \cap B(\alpha, \varepsilon) \subset \overline{\Omega}_0,$$

contradicting $\mathcal{L}^m(\overline{\Omega}_0) = 0$. □

Lemma 2 states that if the exclusion set has Lebesgue measure zero it should be part of the topological boundary of the type set. Assumption 1 is crucial for this result. If it does not hold it is easy to come up with counter-examples even in the one-dimensional case. For examples, see Jullien [17].

Let $\hat{s}$ solve the monopolist's problem without the participation constraint. Note that $\overline{\Omega}_0$ can be expressed as

$$\overline{\Omega}_0 = \{\alpha \in \overline{\Omega} : \hat{s}(\alpha) \leq s_0(\alpha)\}.$$

Now decompose it as

$$\overline{\Omega}_0 = \overline{\Omega}_0^1 \cup \overline{\Omega}_0^2,$$

where $\overline{\Omega}_0^1 = \{\alpha \in \overline{\Omega} : \hat{s}(\alpha) < s_0(\alpha)\}$ and $\overline{\Omega}_0^2 = \{\alpha \in \overline{\Omega} : \hat{s}(\alpha) = s_0(\alpha)\}$.

Lemma 3. $\mathcal{L}^m(\overline{\Omega}_0^2) = 0$.

Proof. If $\overline{\Omega}_0 = \emptyset$ then there is nothing to prove. So assume it is not empty, pick $\alpha, \hat{\alpha} \in \overline{\Omega}_0^2$, with $\hat{\alpha} \le \alpha$ and $\hat{\alpha} \ne \alpha$. Then $u(\hat{\alpha}, x(\hat{\alpha})) = t(x(\hat{\alpha}))$ because $v(\hat{\alpha}, x(\hat{\alpha}), t(x(\hat{\alpha}))) = s_0(\hat{\alpha})$. By Assumption 1,

$$u(\alpha, x(\hat{\alpha})) > u(\hat{\alpha}, x(\hat{\alpha}))$$

and, because v is strictly decreasing in t,

$$s_0(\alpha) = v(\alpha, x(\hat{\alpha}), u(\alpha, x(\hat{\alpha}))) < v(\alpha, x(\hat{\alpha}), t(x(\hat{\alpha}))).$$

But $s_0(\alpha) = v(\alpha, x(\alpha), t(x(\alpha)))$, so the inequality contradicts the optimality of $x(\alpha)$ for type α. Therefore we must have $\alpha \notin \overline{\Omega}_0^2$. The same argument shows that if $\alpha \in \overline{\Omega}_0^2$ and $\hat{\alpha} \le \alpha$, then $\hat{\alpha} \notin \bar{|}w_0^2$, for otherwise α could not be in $\overline{\Omega}_0^2$ in the first place. So for any pair $(\alpha, \hat{\alpha})$ in $\overline{\Omega}_0^2$, we must have $\alpha \nleq \hat{\alpha}$ and $\hat{\alpha} \nleq \alpha$. But this means that $\overline{\Omega}_0^2$ is a *porous set*: for any $\alpha \in \overline{\Omega}_0^2$ and $r > 0$, there is a $r/8$-ball inside of the r-ball centered at α and disjoint from $\overline{\Omega}_0^2$. Porous sets in $\mathbb{R}^m$ have zero $\mathcal{L}^m$-measure.[18] □

Lemma 4. *The surplus function $s : \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous.*

Proof. Take α and α' and a linear path γ connecting these two points. Then

$$|s(\alpha) - s(\alpha')| = \left| \int_{\alpha}^{\alpha'} \nabla v(\gamma(\xi), x(\gamma(\xi)), t(x(\gamma(\xi)))) \cdot \gamma'(\xi) d\xi - \int_{\alpha}^{\alpha'} \nabla s_0(\gamma(\xi)) \cdot \gamma'(\xi) d\xi \right|.$$

As $\overline{\Omega}$ is compact and the functions involved are continuous, we can find a common upper bound $\overline{K}$ for the integrands. As we are integrating over the line connecting α and α' of length $||\alpha - \alpha'||$, the right hand side is bounded by $\overline{K}||\alpha - \alpha'||$, establishing the result. □

Lemma 5. *Under Assumptions 1 and 3, if $\mathcal{L}^m(\overline{\Omega}_0) = 0$ then $\mathcal{H}^{m-1}(\overline{\Omega}_0) = 0$.*

Proof. By Lemma 2, $\overline{\Omega}_0 \subset \partial\Omega$. Because $\mathcal{H}^{m-1}(\partial\Omega \backslash \partial_e\Omega) = 0$, consider $\overline{\Omega}_0 \cap \partial_e\Omega$, which is given by

$$\overline{\Omega}_0 \cap \partial_e\Omega = \bigcup_{i=1}^{\infty} \overline{\Omega}_{0i} \cup (N \cap \overline{\Omega}_0),$$

where

$$\overline{\Omega}_{0i} = \{\alpha \in \overline{\Omega} : g_i(\alpha) = 0, s(\alpha) = 0\}$$

[18] See Zajíček [26] for a reference on porous sets.

for $i \geq 1$. Now Assumptions 1 and 3 ensure that the vectors $\nabla s(\alpha)$ and $\nabla_\alpha g_i(\alpha)$ are linearly independent, for all $i \geq 0$. Hence, by the Implicit Function Theorem,[19] $\overline{\Omega}_{0i}$ is a manifold of dimension at most $m-2$. So $\mathcal{H}^{m-1}(\overline{\Omega}_{0i}) = 0$. Hence

$$\mathcal{H}^{m-1}(\overline{\Omega}_0 \cap \partial_e \Omega) \leq \sum_{i=1}^{\infty} \mathcal{H}^{m-1}(\overline{\Omega}_{0i}) + \mathcal{H}^{m-1}(N \cap \overline{\Omega}_0) = 0,$$

as we wanted to show. □

We are now ready to prove Theorem 1. We will use the Generalized Gauss-Green Theorem, which states that for any Ω with locally finite perimeter in $U \subset \mathbb{R}^m$, and any Lipschitz continuous vector field $\varphi : U \to \mathbb{R}^m$ with compact support in U there is a unique measure theoretic unit outer normal $\tau_\Omega(\alpha)$ such that

$$\int_\Omega \text{div}\varphi d\alpha = \int_U \varphi \cdot \tau_\Omega d\mathcal{H}^{m-1},$$

where

$$\text{div}\varphi = \sum_{k=1}^{m} \frac{\partial \varphi_k}{\partial \alpha_k}$$

is the divergence of the vector field φ.

Proof of Theorem 1. By way of contradiction, assume that $\mathcal{L}^m(\overline{\Omega}_0) = 0$. Let k be a positive integer and denote by π_k the profit obtained by selling to the types in

$$\overline{\Omega}_k = \{\alpha \in \overline{\Omega} : s(\alpha) \leq \frac{1}{k}\}.$$

An implication of Lemma 4 is that $\overline{\Omega}_k$ is a set of locally finite perimeter in the open set U. Indeed, Lipschitz domains are of locally finite perimeter ([21], Proposition 4.5.8.), so the essential boundary of $\overline{\Omega}_k$ is of locally finite $\mathcal{H}^{m-1}$ measure.

Because c is non-negative, we must have

$$\pi_k \leq \int_{\overline{\Omega}_k} t(x(\alpha)) f(\alpha) d\alpha,$$

and from (4) we have

$$\pi_k \leq \int_{\overline{\Omega}_k} u(\alpha, x(\alpha)) f(\alpha) d\alpha.$$

[19]See Theorem M.E.1 in Mas-Colell, Whinston and Green [20]

Observe that with $\mathcal{L}^m(\overline{\Omega}_0) = 0$ we have $\mathcal{L}^m(\overline{\Omega}_k) = \mathcal{L}^m(\overline{\Omega}_k \backslash \overline{\Omega}_0)$, so the envelope condition holds for almost all types in $\overline{\Omega}_k$. Assumption 2 and the implication of the envelope condition (6) yield

$$\pi_k \leq K \int_{\overline{\Omega}_k} \alpha \cdot \nabla s(\alpha) \lambda(\alpha) f(\alpha) d\alpha.$$

Observe that $\lambda : \overline{\Omega} \to \mathbb{R}$ is uniformly continuous because it is continuous and $\overline{\Omega}$ is compact. So λ is the uniform limit of a sequence of Lipschitz continuous functions (Georganopoulos [13].) Hence, for each $e > 0$, let λ_ε be a Lipschitz continuous function such that $\sup_{\alpha \in \overline{\Omega}} ||\lambda_\varepsilon(\alpha) - \lambda(\alpha)|| < \delta$, where $\delta > 0$ is chosen so that

$$\pi_k \leq K \int_{\overline{\Omega}_k} (\alpha \cdot \nabla s(\alpha)) \lambda_\varepsilon(\alpha) f(\alpha) d\alpha + \varepsilon.$$

Applying the Generalized Gauss-Green Theorem to the Lipschitz continuous vector field $\varphi(\alpha) = \alpha s(\alpha) \lambda_\varepsilon(\alpha) f(\alpha)$, and using $\text{div}(\alpha s \lambda_\varepsilon f) = s\text{div}(\alpha \lambda_\varepsilon f) + \lambda_\varepsilon f \alpha \cdot \nabla s$, we get

$$\begin{aligned} \pi_k \leq K &\int_U s(\alpha) \lambda_\varepsilon(\alpha) f(\alpha) (\alpha \cdot \tau_\Omega(\alpha)) d\mathcal{H}^{m-1}(\alpha) \\ - K &\int_{\overline{\Omega}_k} s(\alpha) \text{div}(\alpha \lambda_\varepsilon(\alpha) f(\alpha)) d\alpha + \varepsilon. \end{aligned}$$

The functions s, λ_ε, f, $\alpha \cdot \tau_\Omega$ and $\text{div}(\alpha \lambda_\varepsilon(\alpha) f(\alpha))$ are bounded in $\overline{\Omega}_k$, so we can find a common upper bound B. Because $s(\alpha) \leq \frac{1}{k}$ in $\overline{\Omega}_k$ and $\text{supp}(f) = \overline{\Omega}$, we have

$$\pi_k \leq \frac{1}{k} B(\mathcal{H}^{m-1}(\overline{\Omega}_k) + \mathcal{L}^m(\overline{\Omega}_k)) + \varepsilon.$$

Now consider increasing the tariff by $\frac{1}{k}$. The consumers in the set $\overline{\Omega}_k$ will exit, and π_k will be lost, but each other consumer will pay $\frac{1}{k}$ more. Because the total mass of consumers that exit is bounded by $B\mathcal{L}^m(\overline{\Omega}_k)$, the change in profit is

$$\Delta\pi \geq \frac{1}{k}[(1 - B\mathcal{L}^m(\overline{\Omega}_k)) - B(\mathcal{H}^{m-1}(\overline{\Omega}_k) + \mathcal{L}^m(\overline{\Omega}_k))] - \varepsilon.$$

From Lemma 5, $\mathcal{H}^{m-1}(\overline{\Omega}_0) = 0$, and hence from Lemma 1 we have $\mathcal{L}^m(\overline{\Omega}_k) \to 0$ and $\mathcal{H}^{m-1}(\overline{\Omega}_k) \to 0$, because, by continuity of s and the compact support of f, each $\overline{\Omega}_k$ is compact. But then for large k, we select $\varepsilon_k \in (0, \frac{1-g(k)}{k})$, where $g(k) = B[2\mathcal{L}^m(\overline{\Omega}_k) + \mathcal{H}^{m-1}(\overline{\Omega}_k)]$, so that $\Delta\pi$ is positive, contradicting the optimality of the tariff. □

Moving to Theorem 2, let us keep track of which type space we are dealing with. That is, let us write $\Omega_{0,\beta} = \{\alpha \in \Omega_\beta : s(\alpha; \beta) = 0\}$, where $s(\alpha; \beta)$ is the surplus function obtained by type α when the underlying type space is Ω_β. Likewise, we shall make explicit the dependence of the relevant object on the underlying type space indexed by $\beta \in \mathcal{B}$, viz. $x(\alpha;\beta)$, $x_0(\alpha;\beta)$, etc. Extending $s(\cdot;\beta)$ by continuity to $\partial\Omega_\beta$, let $\overline{\Omega}_{0,\beta} = \{\alpha \in \overline{\Omega}_\beta : s(\alpha;\beta) = 0\}$. We also extend $x(\cdot;\beta)$ by continuity to $\partial\Omega_\beta$.

Note that Lemmas 2, 3, and 4 hold true for any type space Ω_β, $\beta \in \mathcal{B}$. Lemma 5 has to be reformulated as follows:

Lemma 6. *Under Assumptions 1 and 4, if $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$ for all β in some open subset $V \subset \mathcal{B}$, then there exists $\beta' \in V$ such that $\mathcal{H}^{m-1}(\overline{\Omega}_{0,\beta'}) = 0$.*

Proof. By Lemma 2, $\overline{\Omega}_{0,\beta} \subset \partial\Omega_\beta$ for all $\beta \in V$. Because $\mathcal{H}^{m-1}(\partial\Omega_\beta \backslash \partial_e\Omega_\beta) = 0$, consider $\overline{\Omega}_{0,\beta} \cap \partial_e\Omega_\beta$, which is given by

$$\overline{\Omega}_{0,\beta} \cap \partial_e\Omega_\beta = \bigcup_{i=1}^{\infty} \overline{\Omega}_{0i,\beta} \cup (N_\beta \cap \overline{\Omega}_{0,\beta}),$$

where

$$\overline{\Omega}_{0i,\beta} = \{\alpha \in \overline{\Omega}_\beta : g_i(\alpha,\beta) = 0, s(\alpha;\beta) = 0\}$$

for $i \geq 1$. Fix a $\beta \in V$. Assumptions 1 and 4(iii) ensure that there is $\hat{\beta} \in \mathcal{B}$ for which the level sets of $s(\alpha;\beta)$ are transversal to $g_i(\alpha,\hat{\beta})$, for all $i \geq 1$. The Transversality Theorem[20] then implies that, for each i, there exists a set $M_i \in \mathcal{B}$ of full measure such that the level sets of $s(\alpha;\beta)$ are transversal to the level sets of $g_i(\alpha,\beta')$, for all $\beta' \in M_i$. Let $M = \bigcap_{i=1}^{\infty} M_i$, and note that M also has full measure as the measure of its complement is zero. Thus, for all $\beta' \in M$, the level sets of $g_i(\alpha,\beta')$ are transversal to the level sets of $s(\alpha;\beta)$, and hence for any neighborhood of β there exists β' such that the level sets of $g_i(\alpha,\beta')$ are transversal to the level sets of $s(\alpha;\beta)$, for all $i \geq 1$.

Observe that, by compactness of $K_{i,\beta'}$, the set $\{\alpha : g_i(\alpha,\beta') = s(\alpha;\beta) = 0\}$ is finite. Let it be given by $\{\alpha^1, , ..., \alpha^{L_i}\}$. Let $\theta_i^{\ell_i}$ denote the angle between $\nabla_\alpha g_i(\alpha^{\ell_i},\beta')$ and $\nabla s(\alpha^{\ell_i};\beta)$, where $\ell_i \in \{1, ..., L_i\}$. Because of transversality, $\theta_i^{\ell_i} \in (0,\pi)$, for all ℓ_i and i.[21] For each i, let $\varepsilon_i > 0$ be such that $\theta_i^{\ell_i} \in (\varepsilon_i, \pi - \varepsilon_i)$, for $\ell_i = 1, ..., L_i$. Let U_i be an open neighborhood of g_i (in the C^1 topology) such that all functions $\hat{g} \in U_i$, at points α where $\hat{g}(\alpha) = s(\alpha;\beta) = 0$, have the angles between $\nabla\hat{g}(\alpha)$ and $\nabla s(\alpha;\beta)$ within $\varepsilon_i/2$ of the corresponding angles $\theta_i^{\ell_i}$. By part (ii) of Assumption 4, there is a finite collection $\{U_1, ..., U_n\}$ (after relabeling the indices) that covers the family $\{g_i\}_{i\geq 1}$. Let $d = \min\{\varepsilon_1, ..., \varepsilon_n\}/2$ and note that $\theta_i^{\ell_i} \in (d, \pi - d)$ for

[20] Theorem M.E.2 in Mas-Colell, Whinston and Green [20].

[21] Here π denotes the irrational number $\pi = 3.14159...$, and not the profit of the monopolist.

all ℓ_i and i. Observe that, by (5), the optimal tariff t must be continuous, as it is the upper envelope of a family of continuous functions. By continuity of $x(\alpha;\beta)$ in β and continuity of ∇s_0 in β, for all β' sufficiently close to β the angles between $\nabla s(\alpha^{\ell_i};\beta')$ and $\nabla s(\alpha^{\ell_i};\beta)$ are at most $d/4$ apart, for all ℓ_i and all i, as $\nabla s(\alpha;\beta') = \nabla_\alpha v(\alpha, x(\alpha;\beta'), t(x(\alpha;\beta')) - \nabla s_0(\alpha;\beta')$. Further, for β' close enough to β, it must be the case that the angles $\theta_i^{\ell_i}$ are at most $d/4$ apart from the corresponding angles at the new points where $g_i(\alpha,\beta') = s(\alpha;\beta') = 0$. Hence the angles where the level sets of $s(\alpha;\beta')$ and of $g_i(\alpha,\beta')$ intersect must fall in the interval $(d/2, \pi - d/2)$, so the level sets of $s(\alpha;\beta')$ are transversal to the level sets of $g_i(\alpha,\beta')$, for $i \geq 1$. Observe that we can choose β' close enough to β so that $\beta' \in V$.

By the Implicit Function Theorem, for every i, $\mathcal{H}^{m-1}(\overline{\Omega}_{0i,\beta'}) = 0$ as $\overline{\Omega}_{0i,\beta'}$ is a manifold of dimension at most $m-2$. Hence

$$\mathcal{H}^{m-1}(\overline{\Omega}_{0,\beta'} \cap \partial_e \Omega_{\beta'}) \leq \sum_{i=1}^{\infty} \mathcal{H}^{m-1}(\overline{\Omega}_{0i,\beta'}) + \mathcal{H}^{m-1}(N_{\beta'} \cap \overline{\Omega}_{0,\beta'}) = 0,$$

as we wanted to show.[22] □

Lemma 6 provides the basic step in establishing denseness of the set of models where exclusion occurs with positive probability. It is a straightforward application of the standard Transversality Theorem. Nevertheless, it shows that Assumptions 1 and 4 are potent, despite being weak.

Let $\mathcal{E} \subset \mathcal{B}$ be the set of models where the set of excluded consumers has positive measure:

$$\mathcal{E} = \{\beta \in \mathcal{B} : \mathcal{L}^m(\overline{\Omega}_{0,\beta}) > 0\}.$$

Note that the set $\{\alpha \in \overline{\Omega}_{0,\beta} : x(\alpha) \neq x_0(\alpha)\}$ is contained in $\overline{\Omega}^2_{0,\beta}$, so Lemma 3 ensures that $\mathcal{E}$ is indeed the set of models where the set of excluded consumers has positive measure.

Proof of Theorem 2. We shall show that $\mathcal{E}$ is an open and dense subset of $\mathcal{B}$. We start with denseness, using again the Generalized Gauss-Green Theorem.

By way of contradiction, assume that $\mathcal{L}^m(\overline{\Omega}_{0,\beta}) = 0$ for all β in some open set $V \subset \mathcal{B}$. For any natural number k, let $\pi_{k,\beta}$ be the profit obtained by selling to the types in

$$\overline{\Omega}_{k,\beta} = \{\alpha \in \overline{\Omega}_\beta : s(\alpha;\beta) \leq \frac{1}{k}\}.$$

[22] In fact, we established the stronger result that the set of β' such that $\mathcal{H}^{m-1}(\overline{\Omega}_{0,\beta'}) = 0$ is dense in V.

The exact same steps as in the proof of Theorem 1 establish that, for each $\beta \in V$, the change in profit from increasing the tariff by $\frac{1}{k}$ is

$$\Delta\pi_\beta \geq \frac{1}{k}[(1 - B\mathcal{L}^m(\overline{\Omega}_{k,\beta})) - B(\mathcal{H}^{m-1}(\overline{\Omega}_{k,\beta}) + \mathcal{L}^m(\overline{\Omega}_{k,\beta}))] - \varepsilon.$$

From Lemma 6, there exists $\beta' \in V$ with $\mathcal{H}^{m-1}(\overline{\Omega}_{0,\beta'}) = 0$, and hence from Lemma 1 we have $\mathcal{L}^m(\overline{\Omega}_{k,\beta'}) \to 0$ and $\mathcal{H}^{m-1}(\overline{\Omega}_{k,\beta'}) \to 0$, because, by continuity of $s(\cdot;\beta')$ and the compact support of $f(\cdot)$, each $\overline{\Omega}_{k,\beta'}$ is compact. But then for large k, we select $\varepsilon_k \in (0, \frac{1-g(k)}{k})$, where $g(k) = B[2\mathcal{L}^m(\overline{\Omega}_{k,\beta'}) + \mathcal{H}^{m-1}(\overline{\Omega}_{k,\beta'})]$, so that $\Delta\pi_{\beta'}$ is positive, contradicting the optimality of the tariff for the model β'. Therefore, we must have $\mathcal{L}^m(\overline{\Omega}_{0,\beta'}) > 0$. As V was arbitrary, $\mathcal{E}$ is dense, as we wanted to show.

As for openness on $\mathcal{E}$ in $\mathcal{B}$, recall that $\overline{\Omega}_{0,\beta} = \overline{\Omega}^1_{0,\beta} \cup \overline{\Omega}^2_{0,\beta}$. By continuity of $\hat{s}(\cdot;\beta)$ and $s_0(\cdot;\beta)$, $\overline{\Omega}^1_{0,\beta}$ is an open set. By Lemma 3, $\overline{\Omega}^2_{0,\beta}$ has zero Lebesgue measure. $\mathcal{E} \neq \emptyset$ as it is dense, so take $\beta \in \mathcal{E}$, and note that $\overline{\Omega}^1_{0,\beta} \neq \emptyset$. Pick $c > 0$ such that the open set $C_\beta = \{\alpha \in \overline{\Omega}_\beta : \hat{s}(\alpha;\beta) - s_0(\alpha;\beta) < -c\}$ is included in $\overline{\Omega}^1_{0,\beta}$. Pick α in the interior of C_β so that $g_i(\alpha,\beta) > 0$ for all $i \geq 1$. For each i, let $\varepsilon_i > 0$ be such that $g_i(\alpha,\beta) > \varepsilon_i$, and let U_i be an open neighborhood of g_i (in the C^1 topology) such that $\hat{g}(\alpha) > \varepsilon_i/2$ for all functions $\hat{g} \in U_i$. By part (ii) of Assumption 4, there is a finite collection $\{U_1, ..., U_n\}$ (after relabeling of indices) that covers the family $\{g_i\}_{i\geq 1}$. Let $d = \min\{\varepsilon_1, ..., \varepsilon_n\}/2$ and note that $g_i(\alpha,\beta) > d$ for every $i \geq 1$. By the same compactness assumption, we can choose $\varepsilon' > 0$ and $\delta' > 0$ such that $|g_i(\alpha',\beta') - g_i(\alpha,\beta)| < d/2$ whenever $(\alpha',\beta') \in B_{\varepsilon'}(\alpha) \times B_{\delta'}(\beta)$, for every $i \geq 1$. It follows that $B_{\varepsilon'}(\alpha) \subset \Omega_{\beta'}$ for every $\beta' \in B_{\delta'}(\beta)$. Let $h(\alpha;\beta) = \hat{s}(\alpha;\beta) - s_0(\alpha;\beta)$ and pick $\varepsilon \in (0,\varepsilon')$ and $\delta \in (0,\delta')$ such that $|h(\alpha';\beta') - h(\alpha;\beta)| < c/2$ whenever $(\alpha',\beta') \in B_\varepsilon(\alpha) \times B_\delta(\beta)$. It follows that $h(\alpha';\beta') < 0$ for all $(\alpha',\beta') \in B_\varepsilon(\alpha) \times B_\delta(\beta)$. Summing up, $B_\varepsilon(\alpha) \subset \overline{\Omega}_{0,\beta'}$ and hence $\mathcal{L}^m(\overline{\Omega}_{0,\beta'}) > 0$, for every $\beta' \in B_\delta(\beta)$, as we wanted to show. □

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