Ambiguous Pricing and Financial Market Imperfections

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- In such context, securities admit a perfect replication and no-arbitrage condition gives that the price of any security can be computed by its expected value with respect to a unique risk-neutral probability.
- There exists a unique risk-neutral measure (or, a stochastic discount factor) P such that for all security $X : S \to \mathbb{R}$ its price $C(X)$ is given by

$$C(X) = E_P(X).$$

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- - "**Implications of Security Market Data for Models of Dynamic Economies**" by *Hansen and Jagannathan*, JPE 1991;
  - "**Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing**" by *Heaton and Lucas*, JPE 1996;
  - "**Asset Pricing in Economies with Frictions**" by *Luttmer*, Econometrica 1996.

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- In the presence of incompleteness or frictions affecting tradeable securities the no-arbitrage assumption leads to the existence of a set of risk-neutral probabilities in which the correspoding pricing rule is computed by the "largest" expected value.
- There exists a set $\mathcal{Q}$ of probabilities measures (or, a set of stochastic discount factors) such that for all security $X : S \to \mathbb{R}$ its price $C(X)$ is given by

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- Our point is that, in order to perfectly matches the supply and the demand in a context of a competitive financial markets, the Walrasian actioner might consider not only linear pricing rules but also a broader class of pricing rules. In this way, the *tâtonnement* relating to finding a market clearing price give rises to a equilibrium that can incorporates endogenously incompleteness or transactions costs. (The abstract results in Aliprantis, Florenzano and Tourky (JET 2005) and Aliprantis, Tourky and Yannelis (JET 2001) might be important to us)

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- Under this perspective, the case of unambiguous pricing is exactly the context of market perfection.

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- Our main result shows that incompleteness of financial market is related to a extreme case of ambiguous pricing while complete markets with bid-ask spreads is the prevalent case.

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- A bet on the event A is given by the security $A^* : S \to \{0, 1\}$, where $A^*(s) = 1$ iff $s \in A$.
- We denote by $C : \mathbb{R}^S \to \mathbb{R}$ the pricing rule: agents have to pay $C(X)$ units of initial wealth in order to guarantee at least $X(s)$ units of wealth in each state $s \in S$.

Pseudo Pricing Rules

Definition

A pseudo pricing rule $C : \mathbb{R}^S \rightarrow \mathbb{R}$ satisfies [e.g., Chateauneuf et. al. (1996), Jouini (2000), Jouini and Kallal (2001), Castagnoli, Maccheroni and Marinacci (2002)]:

i) C is sublinear, *i.e.*,

$$C(\lambda X) = \lambda C(X), \text{ and}$$

$$C(X+Y) \leq C(X) + C(Y),$$

for all $X, Y \in \mathbb{R}^S$ and all non-negative real number λ;

ii) C is arbitrage free, *i.e.*, $C(X) > 0$ for any nonzero security $X \geq 0$;

iii) C is normalized, *i.e.*, $C(S^*) = 1$;

iv) C is monotonic, *i.e.*, $C(X) \geq C(Y)$ for all $X, Y \in \mathbb{R}^S$ s.t. $X \geq Y$;

v) C is constant additive, *i.e.*, $C(X + kS^*) = C(X) + k$,for all $X \in \mathbb{R}^S$ and all real number k.

Pseudo Pricing Rules Representation

Theorem

- *For any pseudo pricing rule there exists a closed and convex set $\mathcal{Q}$ of probabilities measures, where at least one element is strictly positive, such that for any security X*

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- **A caveat:** Usually, the set of attainable payoffs in a financial economy is generate by a finite set of assets (e.g., Subsection 3.3. in Luttmer (1996)). Hence, the correspoding set of risk-neutral probability has as its closure a **polytope** by a well know result (e.g., Theorem 2.4.6 in Schneider (1993)).

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- For example, assume that $\#S = 3$, for any $\varepsilon > 0$ such that $\mathcal{Q} = B\left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \varepsilon\right) \subset \Delta$, the corresponding pseudo pricing rule can't be related to a (finite) financial market.

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Definition

A mapping $C : \mathbb{R}^S \to \mathbb{R}$ is **additive under finitely convex cone restriction** if there exist a finite collection of convex cone $\{W_l\}_{l=1}^{L}$ with the property that $\cup_l W_l = \mathbb{R}^S$, *span* $\{S^*\} \subset \cap_l W_l$ and given $l \in \{1, ..., L\}$ for all $X, Y \in W_l$

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Theorem

Given a pseudo pricing rule $C : \mathbb{R}^S \to \mathbb{R}$, the corresponding set of probabilities $\mathcal{Q}$ is a polytope if, and only if, C is additive under finitely convex cone restriction.

Can a Pricing Rule Reveals the Underlying Market Structure/Imperfections?

- Given a particular nonlinear pricing rule represented by a set of probabilities, there are many candidates for its underlying type of market structure/imperfections.

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- Given a particular nonlinear pricing rule represented by a set of probabilities, there are many candidates for its underlying type of market structure/imperfections.
- The case frictionless incomplete markets of securities is completely characterized in terms of pricing rules in Araujo, Chateauneuf and Faro (2012).
- Now, we aim to study the case of complete markets with frictions, as well as the general case mixing both possibilities and its consequence for market analysis.

Super-replication Price of Arbitrage-Free Securities Markets

- There is a finite number of assets $X_j \in \mathbb{R}^S$, $0 \leq j \leq m$, with the possibility of bid-ask spreads, which is modeled by a couple of prices for each asset j given by

$$\left(q_j^A, q_j^B\right) \text{ with } q_j^A \geq q_j^B.$$

Also, we suppose that $X_0 = S^* := (1, ..., 1)$ is the *riskless bond* with null bid-ask spread under the price normalization $q_0 = 1$ (*i.e.*, $q_0^A = q_0^B = 1$). Note that we are not allowing bid-ask at liquidation.

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- A portfolio of an agent is identified with a pair of vectors $\left(\theta^A, \theta^B\right) \in \mathbb{R}^{2(m+1)}$, where θ_j^A represents the number of units of asset j **bought** while θ_j^B represents the number of units of asset j **sold** by the agent.

- We recall that an arbitrage opportunity is a portfolio strategy with no cost that yields a strictly positive profit in some states and exposes no loss risk.

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- Formally, a market $\mathcal{M} = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^{m}$ offer *no-arbitrage opportunity* if for any portfolio $\left(\theta^A, \theta^B\right) \in \mathbb{R}_+^{2(m+1)}$,

$$\begin{aligned}
\sum_{j=0}^{m}\left(\theta_j^A - \theta_j^B\right) X_j &> 0 \Rightarrow \sum_{j=0}^{m}\theta_j^A q_j^A - \sum_{j=0}^{m}\theta_j^B q_j^B > 0, \\
\sum_{j=0}^{m}\left(\theta_j^A - \theta_j^B\right) X_j &= 0 \Rightarrow \sum_{j=0}^{m}\theta_j^A q_j^A - \sum_{j=0}^{m}\theta_j^B q_j^B = 0.
\end{aligned}$$

- An important result says that the market $\mathcal{M} = \left\{ X_j, \left(q_j^A, q_j^B \right) \right\}_{j=0}^{m}$ offers no arbitrage opportunity if and only if there exists a (strictly positive) probabilitity $P_0 \in \Delta^+$ such that $q_j^B \leq E_{P_0}(X_j) \leq q_j^A$, $0 \leq j \leq m$.

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- The set

$$\mathcal{Q}_{\mathcal{M}} = \{ P \in \Delta^+ : q_j^B \leq E_P(X_j) \leq q_j^A, \forall j \in \{0, ..., m\} \},$$

is called the set of *risk-neutral probabilities*.

- The pricing rule C generates by the market $\mathcal{M} = \left\{ X_j, \left(q_j^A, q_j^B \right) \right\}_{j=0}^{m}$ is defined by the super-replication price given by, for all $X \in \mathbb{R}^S$

$$C(X) = inf \left\{ \sum_{j=0}^{m} \left(\theta_j^A q_j^A - \theta_j^B q_j^B \right) : \sum_{j=0}^{m} \left(\theta_j^A - \theta_j^B \right) X_j \geq X \right\}.$$

It is worth noticing that for a securities market $\mathcal{M}$ offering no-arbitrage opportunity, the super-replication prices can be also represented by (Jouini and Kallal (1995))

$$C(X) = \sup_{P \in \mathcal{Q}_{\mathcal{M}}} E_P(X), \text{ for all } X \in \mathbb{R}^S.$$

Hence, by taking the closure of the set of risk neutral probabilities $\mathcal{Q} := \overline{\mathcal{Q}_{\mathcal{M}}}$, we have

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X), \text{ for all } X \in \mathbb{R}^S.$$

- Given a financial market $\mathcal{M} = \left\{ X_j, \left(q_j^A, q_j^B \right) \right\}_{j=0}^{m}$, recall that this market is complete if

$$\left\{ \sum_{j=0}^{m} \left(\theta_j^A - \theta_j^B \right) X_j : \left(\theta^A, \theta^B \right) \in \mathbb{R}_+^{2(m+1)} \right\} = \mathbb{R}^S.$$

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- Consider the financial market $\mathcal{M} = \left(S^*, \{s_1\}^* ; (1,1), (1/2, 1) \right)$ and its related pricing rule

$$C(x_1, x_2) = \max_{\alpha \in [1/2, 1]} \{ \alpha x_1 + (1-\alpha) x_2 \} \text{ for all } X = (x_1, x_2).$$

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- (Baccara, Battauz, and Ortu (2006)) **Effective Securities**:
- Long-effective ("$\theta^A > 0$") and short-effective ("$\theta^B > 0$"): *"a security X is long (short)-effective when it is optimal to take a long (short) position in the security X to super-replicate some future cashflow at the minimum cost."*

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- Effectiveness conveys the idea that the security matters for the super-replication capabilities of the investors.

Pricing Rules and Frictionless Securities

- Given a pricing rule $C : \mathbb{R}^S \to \mathbb{R}$, the induced set of **frictionless securities** is defined by

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- Any pricing rule C gives rise a linear subspace of frictionless securities F_C (see ACF 2012).

Pricing Rules and Undominated Securities

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- We note that **all frictionless security** X **is undominated** $(F_C \subset L_C)$; see ACF (2012).

Main Result in Araujo, Chateauneuf and Faro (2012):

Theorem

A pricing rule C is a super-replication price of a frictionless and arbitrage-free securities market including the riskless bond if, and only if, C is a pricing rule satisfying $F_C = L_C$.

Examples

- Consider the *insurance functional* $C : \mathbb{R}^3 \to \mathbb{R}$ (an example of pricing rules in Castagnoli, Maccheroni and Marinacci, 2002) defined by

$$C(X) = \begin{cases} \frac{1}{2}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3, & \text{if } x_1 \geq x_2 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3, & \text{if } x_1 < x_2 \end{cases}.$$

We note that, for all security $X = (x_1, x_2, x_3)$

$$C(X) = \max_{P \in co\left\{\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right)\right\}} E_P(X).$$

It is simple to see that $F_C = \left\{X \in \mathbb{R}^3 : x_1 = x_2\right\}$ and $X = (1, 2, 0) \in L_C$ with bid-ask spread $1/4$. Hence, C is not a super-replication price of a frictionless incomplete market.

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- Can we identify the underlying market revealed by this pricing rule?

Examples

- Given a strictly positive probability $Q \in \Delta$, consider $C : \mathbb{R}^S \to \mathbb{R}$ defined by

$$C(X) = (1 - \varepsilon) E_Q(X) + \varepsilon \max X(S),$$

which we call an *epsilon-contaminated pricing rule*.

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- In this case $F_C = span\{S^*\}$ and $L_C = \mathbb{R}^S$. Hence, C is not a super-replication price of a frictionless securities market.

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- In this case $F_C = span\{S^*\}$ and $L_C = \mathbb{R}^S$. Hence, C is not a super-replication price of a frictionless securities market.
- Can we identify the underlying market revealed by this pricing rule?

- L_C is the pricing rules counterpart of the notion of Effective Securities in the sense that, for any strictly monotone utility U on $\mathbb{R}^S$ and $W > 0$

$$X^* \in \arg \max_{C(X) \leq W} U(X) \Rightarrow X^* \in L_C.$$

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Effective Complete Markets

Theorem

Effective market completeness holds for a pricing rule C if and only if the corresponding extended set of risk neutral probabilies contains only full support probabilities. That is,

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Corollary

If C is a pricing rule related to a incomplete market of securities then there exists a linear pricing rule with non full-support in $\mathcal{Q}$, i.e., $\mathcal{Q} \cap (\Delta^+)^c \neq \emptyset$.

Effective Complete Markets

Theorem

A pricing rule C is a super-replication price of an arbitrage-free financial market $\mathcal{M} = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^{m}$ if and only if the set of probabitilies measures $\mathcal{Q}$ characterizing C is a polytope (i.e., there exists a finite set $\{P_i\}_{i=1}^{n} \subset \Delta$ s.t. $\mathcal{Q} = co\left(\{P_i\}_{i=1}^{n}\right)$.

Theorem

C is pricing rule of a arbitrage-free effective complete market of securities $\mathcal{M} = \left\{ X_j, \left(q_j^A, q_j^B \right) \right\}_{j=0}^{m}$ *if and only if the polytope* $\mathcal{Q}$ *characterizing C is such that* $\mathcal{Q} \subset \Delta^+$, *that is, there exists a finite set of risk neutral probabitilies* $\{P_i\}_{i=1}^{n} \subset \Delta^+$ *such that* $\mathcal{Q} = co\left(\{P_i\}_{i=1}^{m}\right)$.

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- In such polytope representation results of markets with bid-ask spreads we use the results in the Subsection 2.4 of *Schneider, R.* (1993), Convex Bodies: The Brunn-Minkowski Theory.

What about other types of frictions?

- The general message of our results is that **ambiguous pricing rules can always be related to incomplete markets or bid-ask spreads with the strong and central message that the prevalent case is the context of a effective complete market with bid-ask spreads.** But, what we can say about others types of frictions? Our result implies that all frictions like *margin or collateral requirements, bid-ask at liquidation* and *short-sale constraints* can be related to by bid-ask spreads. For instance, it is not hard to show directly that:

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Theorem

Consider an arbitrage-free financial market

$$\mathcal{M} = \left\{ X_j = \left(X_j^A, X_j^B \right), \left(q_j^A, q_j^B \right) \right\}_{j=0}^{m}$$

then there exist a financial market without bid-ask at liquidation $\mathcal{M}' = \left\{ Y_k, \left(p_k^A, p_k^B \right) \right\}_{k=0}^{n}$ *such that* $\mathcal{Q}_{\mathcal{M}} = \mathcal{Q}_{\mathcal{M}'}$.

Remark on the Density of Effective Complete Markets.

- Consider the family of all convex bodies of Δ denoted by $\mathcal{K}(\Delta) := \{\mathcal{Q} \subset \cdot : \mathcal{Q}$ is nonempty, compact, convex$\}$ endowed with the Hausdorff distance d_H.

[1] See also Gruber (1983), "Approximation of convex bodies" and Gruber and Kenderov (1982) "Approximation of convex bodies by polytopes."

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- $(\mathcal{K}(\Delta), d_H)$ is a separable Banach space and the set of polytopes is dense on it. See, for instance, Schneider (1993), Theorem 2.4.14[1].
- *We can approximate any pseudo pricing rule by a pricing rule of an effective complete market of securities.*

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Complete markets with uniform bid-ask spreads

Definition

We say that a financial market $\mathcal{M} = \left\{X_j, \left(q_j^A, q_j^B\right)\right\}_{j=0}^{m}$ with a frictionless bond satisfies the uniform bid-ask condition if it is complete and there exist $\varepsilon > 0$ such that:

i) Given an Arrow security $\{s\}^*$, its bid-ask spread is given by

$$q_s^A - q_s^B = \varepsilon;$$

ii) Selling all Arrow securities and buying the bond generate a cost $\varepsilon > 0$, *i.e.*,

$$1 - \sum_{s=1}^{\#S} q_s^B = \varepsilon.$$

Complete markets with uniform bid-ask spreads

Theorem

A pricing rule C is the super-replication price of a effective complete securities market with **uniform bid-ask spreads** *iff there are a strictly positive probability P and a constant $\varepsilon \in (0,1)$ such that for all security X*

$$C(X) = (1-\varepsilon) E_P(X) + \varepsilon \max_{s \in S} X(s).$$

Complete markets of Arrow securities with bid-ask spreads

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Complete markets of Arrow securities with bid-ask spreads

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- There exists a collection of pairs $\{(a_s, b_s)\}_{s \in S}$ with $0 < b_s \leq a_s < 1$ and $\Sigma b_s \leq 1 \leq \Sigma a_s$ such that for any event $E \subset S$

$$v(E) := \max \left\{ 1 - \sum_{s \in E^c} b_s, \sum_{s \in E} a_s \right\},$$

and the pricing rule is given by the Choquet integral w.r.t. v.

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  2. For an economy with strong optimistic investor (nonconvex preferences generated by a special class of concave capacities) there exists an equilibrium with nonlinear prices revealing an endogenous incomplete market (In some exceptional examples we have identified linear prices in such economies). Araujo, Chateauneuf, Faro and Novinski *working in progress*.

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  3. If we allow nonlinear prices, we enlarge the set of equilibria in an Economy with expected utility agents (Aliprantis et. at. 2001, 2005 JET). One open point is how to select the equilibrium prices by a reasonable process, for instance, when the Knightian Market Maker chooses the equilibrium prices that maximizes the revenue from bid-ask spreads.